Boson realisation of symplectic algebras

1985 J. Phys. A: Math. Gen. 18 L1

(http://iopscience.iop.org/0305-4470/18/1/001)

J. Phys. A: Math. Gen. **18** (1985) L1–L6. Printed in Great Britain

LETTER TO THE EDITOR

Boson realisation of symplectic algebras

M Moshinsky†

Instituto de Física, UNAM, Apdo Postal 20-364, México, DF, 01000 México

Received 11 October 1984

Abstract. The boson realisation of the two-dimensional symplectic Lie algebra sp(2) for a given irreducible representation (irrep) of the Sp(2) group has been known for a long time. More recently the boson realisation of the $2d$-dimensional symplectic Lie algebras sp($2d$) have been derived for particular irreps of the Sp($2d$) group. In this letter we outline the corresponding result for an arbitrary irrep of Sp($2d$).

As early as 1940 Holstein and Primakoff obtained a realisation of the su(2) Lie algebra in terms of boson creation and annihilation operators for a given value of the Casimir operator, i.e. for a definite irreducible representation (irrep) of the SU(2) group. It is easy to extend this realisation (Mello and Moshinsky 1976, Deenen and Quesne 1982a) to the su(1, 1) ≃ sp(2) Lie algebra for a definite irrep of the Sp(2) group. This simple result immediately suggests the following question: Is it possible to obtain the boson realisation of a sp(2d) Lie algebra for an arbitrary irrep of the Sp($2d$) group when d is any integer?

One answer to this question has already been given by Rowe (1984), but this letter is based on the point of view developed by Castaños *et al* (1984b), who gave a complete discussion of this problem when $d=2$, i.e. sp(4). As this case already has the main features of the general problem, we proceed to extend the results to an arbitrary sp($2d$).

We start by expressing the generators of the sp($2d$) Lie algebra in terms of creation η_{is} and annihilation ξ_{is} operators of a system of n particles, which are associated with the index $s=1, 2, \ldots, n$, in a d-dimensional harmonic oscillator potential for which the component index is $i=1, 2, \ldots, d$. The generators of sp($2d$) are then (Deenen and Quesne 1982b, Castaños *et al* 1984b)

$$B^+_{ij}=\sum_{s=1}^{n}\eta_{is}\eta_{js}, \tag{1a}$$

$$C_{ij}=\tfrac{1}{2}\sum_{s=1}^{n}(\eta_{is}\xi_{js}+\xi_{js}\eta_{is})=\sum_{s=1}^{n}\eta_{is}\xi_{js}+\tfrac{1}{2}n\delta_{ij}, \tag{1b}$$

$$B_{ij}=\sum_{s=1}^{n}\xi_{is}\xi_{js}, \tag{1c}$$

where in ($1b$) we used the commutation rule $[\xi_{jt}, \eta_{is}]=\delta_{ij}\delta_{st}$. As is customary (Moshinsky 1968) we can divide the set of operators (1) into raising, weight and lowering

† Member of El Colegio Nacional and Instituto Nacional de Investigaciones Nucleares.

generators given by

$$B^+_{ij}; \qquad C_{ij} \text{ with } i<j, \tag{2a}$$

$$C_{ii}, \qquad i=1,2,\ldots,d, \tag{2b}$$

$$B_{ij}; \qquad C_{ij} \text{ with } i>j. \tag{2c}$$

The lowest weight (LW) state denoted by the ket$|\mathrm{LW}\rangle$ then satisfies the equations

$$B_{ij}|\mathrm{LW}\rangle=0, \qquad C_{ij}|\mathrm{LW}\rangle=0 \qquad \text{for } i>j; \tag{3a, b}$$

$$C_{ii}|\mathrm{LW}\rangle=(\omega_i+\tfrac{1}{2}n)|\mathrm{LW}\rangle, \tag{3c}$$

where we note that in (3c) the ω_i are integer eigenvalues of the number operators $\Sigma_{s=1}^{n}\eta_{is}\xi_{is}$, $i=1,2,\ldots,d$, which furthermore from (3b) satisfy the inequality $0\leq\omega_1\leq\omega_2\ldots\leq\omega_d$. The eigenvalues of C_{ii} characterise thus the irrep of Sp(2d) and they can be put in the order

$$[\omega_1+\tfrac{1}{2}n, \omega_2+\tfrac{1}{2}n, \ldots, \omega_d+\tfrac{1}{2}n]. \tag{4}$$

The question raised at the end of the first paragraph refers now to our knowledge of the boson realisation of the sp(2d) Lie algebra associated with the irrep (4) of the Sp(2d) group.

When all the ω_i in (4) are equal an answer was outlined by the author (Moshinsky 1982), while a full and detailed analysis of the problem by an independent procedure was given by Deenen and Quesne (1982b). For $d=2$ but a general irrep, i.e. $\omega_1\neq\omega_2$ the problem was discussed by Castaños *et al* (1984b). In this note we outline the solution for arbitrary d and any irrep of the type (4).

We start by indicating, as was discussed in other publications (Deenen and Quesne 1983a, Moshinsky 1984, Moshinsky *et al* 1984, Rowe *et al* 1984), that we not only need now the symmetric boson operators $b^+_{ij}=b^+_{ji}$, $b_{ij}=b_{ji}$, satisfying the commutation relations

$$[b^+_{ij}, b^+_{i'j'}]=0, \qquad [b_{ij}, b_{i'j'}]=0, \qquad [b_{ij}, b^+_{i'j'}]=\delta_{ii'}\delta_{jj'}+\delta_{ij'}\delta_{ji'}, \tag{5a, b, c}$$

but also an independent set of generators of a Lie algebra u(d), which we designate by S_{ij}, and that have the commutation relations

$$[S_{ij}, b^+_{i'j'}]=0, \qquad [S_{ij}, b_{i'j'}]=0, \qquad [S_{ij}, S_{i'j'}]=S_{ij'}\delta_{i'j}-S_{i'j}\delta_{ij'}. \tag{5d, e, f}$$

We require also that b^+_{ij}, b_{ij} are Hermitian conjugate to each other and that $S^+_{ij}=S_{ji}$. Clearly the b^+_{ij}, b_{ij}, S_{ij} are the generators of a direct sum of a Weyl Lie algebra in $\frac{1}{2}d(d+1)$ dimensions and a unitary Lie algebra in d dimensions, i.e. $w[d(d+1)/2]\oplus$ u(d). We shall frequently use a matrix notation for the operators mentioned above, e.g.

$$\boldsymbol{B}^+=\|B^+_{ij}\|, \qquad \boldsymbol{b}^+=\|b^+_{ij}\|, \qquad \boldsymbol{S}=\|S_{ij}\|, \qquad \text{etc.} \tag{6a, b, c}$$

We now define the matrix $\boldsymbol{L}=\|L_{ij}\|$ by the product

$$\boldsymbol{L}=\boldsymbol{b}^+\boldsymbol{b}, \tag{7}$$

as well as the matrix $\boldsymbol{J}=\|J_{ij}\|$ by the sum

$$\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}. \tag{8}$$

From (7) and (5) we see immediately that the commutation relations of the elements of the matrix $\boldsymbol{L}$ are

$$[L_{ij}, L_{i'j'}] = L_{ij'}\delta_{i'j} - L_{i'j}\delta_{ij'}, \tag{9}$$

and similarly for the elements of J, indicating that these matrices are generators of u(d) Lie algebras.

We would like now to express B^+_{ij}, B_{ij}, C_{ij} of (1) in terms b^+_{ij}, b_{ij}, S_{ij} in such a way that when the latter satisfy the commutation rules (5), the former have the ones associated with an sp(2d) Lie algebra, i.e.

$$[B_{ij}, B_{i'j'}] = 0, \qquad [B^+_{ij}, B^+_{i'j'}] = 0, \tag{10a, b}$$

$$[C_{ij}, B^+_{i'j'}] = B^+_{ij'}\delta_{ji'} + B^+_{ii'}\delta_{jj'}, \tag{10c}$$

$$[C_{ij}, B_{i'j'}] = -B_{jj'}\delta_{ii'} - B_{ji'}\delta_{ij'}, \tag{10d}$$

$$[B_{ij}, B^+_{i'j'}] = C_{j'j}\delta_{ii'} + C_{i'j}\delta_{ij'} + C_{j'i}\delta\delta_{ji'} + C_{ii'}\delta_{jj'}, \tag{10e}$$

$$[C_{ij}, C_{i'j'}] = C_{ij'}\delta_{i'j} - C_{i'j}\delta_{ij'}. \tag{10f}$$

To begin with if we write

$$C_{ij} = J_{ij} \tag{11}$$

as suggested in previous publications (Deenen and Quesne 1983a, b, Rowe *et al* 1984, Moshinsky 1984) it is clear that (10f) is satisfied. For B^+_{ij} we note that with respect to the U(d) subgroup of Sp(2d) whose generators are the C_{ij} it corresponds to the irrep given by the partition [2]. Similarly the b^+_{ij} correspond to the partition [2] for the U(d) group whose generators are the J_{ij} of (8). To express then B^+_{ij} in terms of b^+_{ij} we proceed as follows. We consider the operators that are traces (Tr) of the matrices

$$\Phi_{rp} \equiv \mathrm{Tr}(\boldsymbol{L}^{r-p}\boldsymbol{S}^p), \qquad 0 \leqslant p \leqslant r-1, 1 \leqslant r \leqslant d, \tag{12}$$

of which clearly we have $\frac{1}{2}d(d+1)$. These operators commute with the J_{ij} as can be easily seen by induction and thus are invariants of U(d). Clearly then the commutator

$$[\Phi_{rp}, b^+_{ij}] \tag{13}$$

is also characterised by the irrep [2] of U(d) and thus we can write

$$\boldsymbol{B}^+ = \sum_{r=1}^{d} \sum_{p=0}^{r-1} [\Phi_{rp}, \boldsymbol{b}^+] F^{rp} \tag{14}$$

where the F^{rp} are functions of b^+_{ij}, b_{ij}, S_{ij} that are invariant under the U(d) group (Castaños *et al* 1984b). To get $\boldsymbol{B}$ in terms of $\boldsymbol{b}$ we just have to take the Hermitian conjugate of (14).

The question is now how to determine the F^{rp}. For this we need the matrix elements for $\boldsymbol{B}^+$ of (14) with respect to an appropriate set of boson states that include the intrinsic states related to the unitary group in d-dimensions associated with the S_{ij}. For the bosons the states are given by polynomials in the b^+_{ij} acting on the vacuum state $|0)$. These polynomials can be characterised by an irrep $[\lambda_1 \ldots \lambda_d]$ of U(d) in which all the λ's are even (Moshinsky 1982, Deenen and Quesne 1982b), i.e.

$$P_{[\lambda_1 \ldots \lambda_d]}(b^+_{ij})|0). \tag{15}$$

The intrinsic states are eigenstates of the Casimir operators $\mathrm{Tr}(\boldsymbol{S}^r)$, $r = 1, \ldots, d$, of the

d-dimensional unitary group and thus can be characterised by the partition in the ket

$$|[\omega_1\omega_2 \ldots \omega_d]). \tag{16}$$

The full state is the direct product of (15) and (16) (Deenen and Quesne 1984, Rosensteel and Rowe 1983, Rowe *et al* 1984, Moshinsky *et al* 1984) and it is convenient to couple it to a definite irrep of U(d), i.e.

$$|[\lambda_1 \ldots \lambda_d][\omega_1 \ldots \omega_d]\Omega[h_1 \ldots h_d]) = [P_{[\lambda_1 \ldots \lambda_d]}(b_{ij}^+)|0) \times |[\omega_1 \ldots \omega_d])]_{\Omega[h_1 \ldots h_d]} \tag{17}$$

where the Ω are the set of multiplicity indices (Moshinsky 1963) required to characterise the Clebsch–Gordan coefficient of U(d).

We now consider the reduced matrix elements of the right-hand side of (14) with respect to the states (17). We have then in it the matrix elements

$$\begin{aligned}([\lambda_1' \ldots \lambda_d'][\omega_1 \ldots \omega_d]\Omega'[h_1 \ldots h_d]\|F^{rp}\|[\lambda_1 \ldots \lambda_d][\omega_1 \ldots \omega_d]\Omega[h_1 \ldots h_d]) \\ \equiv F^{rp}_{\Omega'[\lambda_1' \ldots \lambda_d'],\Omega[\lambda_1 \ldots \lambda_d]}([h_1 \ldots h_d],[\omega_1 \ldots \omega_d])\end{aligned} \tag{18}$$

as well as those of

$$\Phi_{rp} = \sum_{i,j=1}^{d} (\boldsymbol{L}^{r-p})_{ij}(\boldsymbol{S}^p)_{ji} \tag{19}$$

and b_{ij}^+, where the latter can be obtained with the help of Racah coefficients (Rosensteel and Rowe 1983) of U(d) and well known (Moshinsky 1968) reduced matrix elements of $(\boldsymbol{L}^{r-p})_{ij}$, b_{ij}^+ with respect to the states (15) and of $(\boldsymbol{S}^p)_{ji}$ with respect to the states (16). Thus if the reduced matrix element of $\boldsymbol{B}^+$ on the left-hand side of (14) is known independently we get a set of linear equations for the unknowns (18).

Note that the number of unknowns of the type (18) is given by

$$[\tfrac{1}{2}\mathrm{d}(d+1)]q^2 \tag{20}$$

where q is the number of possibilities for $\Omega[\lambda_1 \ldots \lambda_d]$ consistent with fixed $[h_1 \ldots h_d]$, $[\omega_1 \ldots \omega_d]$, while $\frac{1}{2}d(d+1)$ is the corresponding number for r, p.

The number of matrix elements of $\boldsymbol{B}^+$ for fixed $[h_1' \ldots h_d']$, $[h_1 \ldots h_d]$ is also related to q^2, but besides

$$[h_1' \ldots h_d'] = [h_1 \ldots h_j + 2 \ldots h_d] \qquad \text{or} \qquad [h_1 \ldots h_i + 1 \ldots h_j + 1 \ldots h_d] \tag{21}$$

for all values $1 \leq i < j \leq d$ which gives $\frac{1}{2}d(d+1)$ other possibilities. Thus the number of equations is also given by (20) and equals the number of unknowns which allows their complete determination.

Our problem is now to find an independent procedure of obtaining the reduced matrix elements of the operator $\boldsymbol{B}^+$ with respect to the states (17). Fortunately this has been developed recently (Rowe *et al* 1984, Deenen and Quesne 1984, Moshinsky *et al* 1984b, Castaños *et al* 1984a) in connection with efforts for determing the matrix elements of the generators of sp($2d$) Lie algebra for the states characterised by irreps of the chain of groups Sp($2d$) $\supset$ U(d). In these developments the generator $\boldsymbol{B}^+$ is written as

$$b^+ = \boldsymbol{K}\boldsymbol{b}^+\boldsymbol{K}^{-1} \tag{22}$$

where K is an Hermitian operator invariant under u(d), i.e. $[J_{ij}, K] = 0$. The operator

K satisfies the equation (Moshinsky *et al* 1984, Rowe *et al* 1984, Deenen and Quesne 1984)

$$\boldsymbol{b}K^2 = K^2(\boldsymbol{bS} + \tilde{\boldsymbol{S}}\boldsymbol{b} + \boldsymbol{bb}^+\boldsymbol{b} - (d+1)\boldsymbol{b}), \tag{23}$$

where $\tilde{\boldsymbol{S}}$ is the transposed of the matrix $\boldsymbol{S}$. As the matrix elements of $\boldsymbol{b}$, $\boldsymbol{b}^+$, $\boldsymbol{S}$ with respect to the states (17) are well known, the equation (23) provides a recursion relation for the matrix elements of K^2 with respect to these states. From the matrix K^2 one can get, at least numerically, the matrix elements of K, K^{-1} with respect to the states (17) and thus, substituting in (22), get an independent procedure for determining the matrix elements of $\boldsymbol{B}^+$.

We see thus that, at least numerically, one can get the matrix elements of the operators $F^{\prime p}$ appearing in (14) and thus have a boson realisation of the generators of the sp$(2d)$ Lie algebra. If the matrix elements of K can be obtained from those of K^2 analytically—which implies that q of (20) must be small enough so that the $q\times q$ matrix K for fixed $[h_1 \dots h_d]$, $[\omega_1 \dots \omega_d]$ gives a secular equation

$$\det(K - \lambda I) = 0 \tag{24}$$

of degree smaller than five—then the matrices of the $F^{\prime p}$ operators can also be obtained analytically and they can in turn be written as functions of $\boldsymbol{b}^+$, $\boldsymbol{b}$, $\boldsymbol{S}$ that are invariant under the u(d) Lie algebra whose generators are the $\boldsymbol{J}$.

In these cases there is then an explicit operator relation between the generators (1) of sp$(2d)$ and those of $w[\frac{1}{2}d(d+1)]\oplus$su$(d)$, i.e. the $\boldsymbol{b}^+$, $\boldsymbol{b}$, $\boldsymbol{S}$. This relation is trivial to determine (Mello and Moshinsky 1976, Deenen and Quesne 1982a) for $d=1$ and was discussed explicitly (Castaños *et al* 1984b) for some special cases when $d=2$, as well as for arbitrary d where in the irrep (4) of Sp$(2d)$ all the ω_i are equal (Deenen and Quesne 1982b).

The present analysis thus generalises to any d and arbitrary irrep (4) of Sp$(2d)$ the particular cases considered previously.

The author would like to thank O Castaños and E Chacón in México and J Deenen and C Quesne in Brussels for many discussions on the subject of the present letter. He would like also to acknowledge a recent preprint of Deenen and Quesne 'Boson representations of the real symplectic group and their applications' in which a different boson realisation is presented.

References

Castaños O, Chacón E and Moshinsky M 1984a *J. Math. Phys.* **25** 1211
Castaños O, Chacón E, Moshinsky M and Quesne C 1984b *J. Math. Phys.* submitted for publication
Deenen J and Quesne C 1982a *J. Math. Phys.* **23** 878
—— 1982b *J. Math. Phys.* **23** 2004
—— 1983a *J. Phys. A: Math. Gen.* **16** 2095
—— 1983b *Lecture Notes in Physics* **180** (Berlin: Springer) p 444
—— 1984 *J. Math. Phys.* **25** 2354
Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
Mello P A and Moshinsky M 1976 *J. Math. Phys.* **16** 2017
Moshinsky M 1963 *J. Math. Phys.* **4** 1128
—— 1968 *Group Theory and the Many Body Problem* (New York: Gordon and Breach)
—— 1982 *Physica* **114A** 322
—— 1984a *Nucl. Phys.* A **421** 81

Moshinsky M, Chacón E, Castaños O 1984b *Constrained bosons for collective states in open shell nuclei, Proc. XIII Colloquium on Group Theoretical Methods in Physics, University of Maryland 1983* (Singapore: World Scientific)
Rosensteel G and Rowe D J 1983 *J. Math. Phys.* **24** 2461
Rowe D J, Rosensteel G and Carr R 1984 *J. Phys. A: Math. Gen.* **17** L399
Rowe D J 1984 *J. Math. Phys.* **25** 2662